



CLASSICAL ELECTROMAGNETISM

SECOND EDITION

Jerrold Franklin

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Jerrold Franklin

TEMPLE UNIVERSITY

Dover Publications, Inc.
Mineola, New York

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Bibliographical Note

This Dover edition, first published in 2017, is a revised edition of the work first published by Pearson / Addison-Wesley in 2005.

Library of Congress Cataloging-in-Publication Data

Names: Franklin, Jerrold, author.
Title: Classical electromagnetism / Jerrold Franklin (Temple University).
Description: Second edition. | Mineola, New York : Dover Publications, Inc., [2017] | Includes bibliographical references.
Identifiers: LCCN 2017031035 | ISBN 9780486813714 | ISBN 0486813711
Subjects: LCSH: Electromagnetism—Textbooks.
Classification: LCC QC760 .F74 2017 | DDC 537—dc23
LC record available at <https://lcn.loc.gov/2017031035>

Manufactured in the United States by LSC Communications
81371101 2017
www.doverpublications.com

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Preface to the second edition

This second edition includes a number of improvements to the first edition in clarifying and extending a number of the derivations, but it serves the same purpose of providing a good learning experience for students. My intention has been to write a comprehensive textbook for the first-year electromagnetism course. It could also serve as supplementary reading for graduate students to help them achieve a better understanding of the subject matter, or for advanced undergraduate students who want to go into more detail than is generally covered in the undergraduate course. I hope it will prove to be helpful for students, and will also be a good textbook from which professors can teach.

Preface to the first edition

This text is designed for first-year graduate students who generally will have had at least one, and probably two, earlier undergraduate courses in Electromagnetism (EM). Because the text starts each topic at a fundamental level and works up, it could also be used by a good student with little prior knowledge of EM, but enough background in mathematics to feel comfortable.

The book tells the story of EM, from its 19th-century beginnings to its present place, somewhere in the 21st century. It starts with simple expositions of Coulomb's law and the magnetic law of Biot-Savart. At each stage in the development it demonstrates how the subject could be more fundamentally defined. This continues through the laws of Gauss and Ampere, to the unifying partial differential equations of Maxwell, and the ensuing electromagnetic radiation. Then it is shown how the principles of Special Relativity *require* the unification of Electricity and Magnetism that had been achieved earlier by Maxwell, and extends this to the unification of Space and Time. In the last chapter, the presentation of Electromagnetism as a Quantum Gauge Theory demonstrates that EM is a manifestation of the general principle of Local Gauge Invariance in Quantum Mechanics. A bit more Quantum Mechanics is introduced to show that Classical Electromagnetism is a limiting form of Quantum Electrodynamics (QED).

As an afterword, it is shown how the generalization of Gauge Invariance to coupled fields leads to the unification of the Electromagnetic and Weak Interactions, and the conjectured unification of these interactions with the Strong Interaction (Grand Unified Theory), ultimately completing the unification of physics started by Maxwell in the 19th century.

The mathematics is completely unified with the physics at each stage in the text. There are no separate mathematical appendices or flyleaf formulas to

memorize. The appropriate mathematics is learned more easily in the context of a real physics application. The mathematical concepts are developed as they are needed for the physics. This makes the learning process for both math and physics more natural and, I hope, more interesting. As with the physics, the mathematics is introduced first on a basic level, but ends up at the high level needed for a good development of the physics of EM.

My own Electromagnetism was learned first from Sears 1st Edition, and then from Panofsky & Phillips. (Information for each book is given in the Bibliography.) Over the years I have used P & P, Jackson, and Good & Nelson as texts for the course I taught. The remarkably interesting books by Mel Schwartz and Landau & Lifschitz were also used for reference by me and my students. Consciously or subconsciously, much of the material in this text draws on those texts (and also on Maxwell's own remarkable 1873 *Treatise*). But it is not an amalgam. You will find many new treatments and new insights. I have tried to write a new text that is enjoyable to learn from, and to teach from, while maintaining a high level of rigor and reaching a deep level of understanding.

The chapters should not all be covered at the same rate. The earlier chapters, especially chapters 1-4, are somewhat of a review and could be covered more quickly. But they are important, with some new ways of looking at old EM. A reasonable break in a one year course would be to complete at least chapters 1-8 in the first semester. It would also be reasonable to go as far as section 9.5 (Magnetic Energy).

A word must be added about units. The two texts I learned my early EM from, Sears 1st Edition (a pioneer in SI, then called Giorgi units) and Panofsky and Phillips, each used Systeme Internationale (SI) units. What I learned was the roadblock that SI units place in the unification of Electricity with Magnetism. (A fuller discussion is in the last section of this book.) I know this book goes against the trend for the use of SI units in all new textbooks, and thank my publisher for permitting it. I hope that my book will go some way against this stream, at least for advanced texts. I use Gaussian units for most of the book, then dispense with the conversion constant c after Relativity, finally getting to natural units in the last chapter. This evolution of units follows the course of unification of Electricity with Magnetism, then with the Weak Interaction, and maybe more. My recommendation for numerical calculations is to use Gaussian (or natural) units throughout any calculation (putting SI quantities into Gaussian using Appendix A). Then, at the end, Gaussian quantities can be put into SI units if desired.

The use of SI units beclouds the obvious connections between the \mathbf{E} and \mathbf{D} fields, and between the \mathbf{B} and \mathbf{H} fields, as well as precluding a simple relativistic unification of the \mathbf{E} and \mathbf{B} fields. Beyond this, SI units make the extension of Classical Electromagnetism to QED, and the unification with Weak Interactions

particularly unwieldy. In fact, SI units fly in the face of all the advances in the unification of physics of the past 150 years. I should also mention the introduction (in SI units) of two misnamed constants (ϵ_0 and μ_0) that have no physical meaning and serve only to complicate EM for beginning students as well as working physicists. Enough about units. I hope you enjoy and learn from my book.

Jerrold Franklin

Chapter 1

Foundations of Electrostatics

1.1 Coulomb's Law

Historically, the quantitative study of electrostatics began in 1784 with **Coulomb's law**, which is illustrated in Fig. 1.1. This law states that the electric force between two point charges is inversely proportional to the square of the distance between them and directly proportional to the product of the charges, with the direction of the force being along the straight line connecting the two charges. In these respects, the Coulomb force between two point charges is similar to the gravitational force between two point masses. An important difference between the two force laws is that the electric charge comes in two signs, with the force between like charges being repulsive, and that between two opposite charges being attractive.

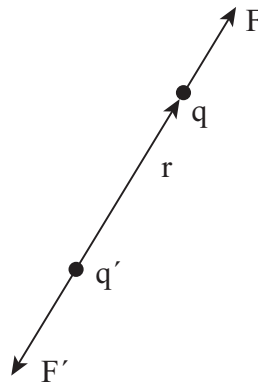


Figure 1.1: Coulomb's law for the force between two charges.

Coulomb's law can be written

$$\mathbf{F} = \frac{kqq'\hat{\mathbf{r}}}{r^2}, \quad (1.1)$$

giving the force on charge q due to charge q' in terms of the unit vector $\hat{\mathbf{r}}$ that specifies the direction from q' to q . The unit vector $\hat{\mathbf{r}}$ is a dimensionless vector defined as a vector divided by its magnitude, $\hat{\mathbf{r}} = \mathbf{r}/|r|$, so Coulomb's law can also be written as

$$\mathbf{F} = \frac{kqq'\mathbf{r}}{r^3}. \quad (1.2)$$

This form of Coulomb's law is more useful in vector operations.

Because Coulomb's law is a proportionality, a constant k is included in Eqs. (1.1) and (1.2). This constant can be chosen to define the unit of electric charge. Unfortunately, several different definitions have been used for the unit of charge, and some care is required in treating the units consistently. We take time now to discuss some of the different systems.

The simplest choice, in terms of Coulomb's law, is to set $k=1$, and use Coulomb's law to define the unit of electric charge. This leads to the electrostatic unit (**esu**) of charge called the **statcoulomb**, which can be defined in words by **the electrostatic force between two charges, each of one statcoulomb, a distance one centimeter apart, is one dyne**. More simply stated, if all distances and forces are in cgs (centimeter-gram-second) units, then the charge in Coulomb's law (with $k=1$) is in statcoulombs.

Another choice for the definition of the unit of electric charge uses MKS (meter-kilogram-second) units in Coulomb's law with the constant k being given by $k \simeq 9 \times 10^9$. This defines the unit of charge called the **coulomb**, which could be defined in words by **the electrostatic force between two charges, each of one coulomb, a distance one meter apart, is 9×10^9 Newtons**.

The coulomb is somewhat more familiar than the statcoulomb because the common unit of current, the ampere, is defined as one coulomb per second. Possibly for this reason, the coulomb, and the form of Coulomb's law with $k \simeq 9 \times 10^9$ was adopted as part of the Systeme International (SI) system of units, and has gained almost universal usage in elementary physics textbooks. However, the study and understanding of electrostatics is considerably simpler using esu units with $k=1$. The SI system is also particularly awkward to use for relativistic or quantum formulations of electromagnetism. For this reason, we will consistently use esu units as part of what is called the **Gaussian system** of units relating the esu and emu systems of units, which will be introduced in Chapter 7. Conversions between the Gaussian system and the SI system are given in Appendix A.

In the SI system, Coulomb's law is further complicated by introducing a new quantity ϵ_o defined by

$$k = \frac{1}{4\pi\epsilon_o}. \quad (1.3)$$

The use of the $1/4\pi$ is said to “rationalize” the units because it makes some later equations (such as Gauss’s law) simpler. The constant ϵ_o is sometimes called “the permittivity of free space.” This terminology is unfortunate because, in the theory of Quantum Electrodynamics (QED), which is the foundation theory for Classical Electromagnetism, a frequency dependent permittivity does arise (vacuum polarization) that has nothing to do with ϵ_o .

The unit of charge in the SI system is related to the esu unit by

$$1 \text{ coulomb} = 3 \times 10^9 \text{ statcoulombs.} \quad (1.4)$$

The number 3×10^9 relating the statcoulomb to the coulomb is related to a constant c , and the number 9×10^9 for the constant k is related to c^2 . The constant c has dimensions of velocity, and was originally introduced for consistency between electric and magnetic phenomena. Some years after its introduction, Maxwell showed that the constant c was, in fact, the speed of light in vacuum (see Chapter 10). Herman Minkowski then showed in Einstein’s theory of Special Relativity (see Chapter 14) that space and time just referred to different directions in a completely symmetric space-time manifold. This makes c the conversion constant between the space axes and the time axis (just like the conversion between miles and feet in an American topographical map). This means that c is no longer a constant to be measured, but a specified number used to define the meter in terms of the second. This is the modern definition of the meter, which is defined so that light travels exactly 299,792,458 meters in one second. In cgs units, then

$$c = 2.99792458 \times 10^{10} \text{ cm/sec,} \quad (1.5)$$

which is the number we will use in this text.

The defined value for c is very close to 3×10^{10} in magnitude (equal to three significant figures), and we will generally use the value 3 for conversions, with the understanding that the more accurate value could be used if greater accuracy were desired. (Whenever the numbers 3 or 9 appear in conversion equations, the more accurate value could be substituted.) The conversion numbers are also changed by various powers of 10 related to the difference between cgs and MKS units, as well as a mismatch in relating the ampere to the emu unit of current, the abampere. (Ten amperes equal one abampere.) The esu unit of charge and the emu unit of charge are related by

$$1 \text{ abcoulomb} = c \text{ statcoulomb,} \quad (1.6)$$

which is the origin of Eq. (1.4) when the powers of 10 are adjusted appropriately.

As an example of the connection between the units, the magnitude of the charge on an electron is given by

$$e = 4.80 \times 10^{-10} \text{ statcoulomb,} \quad (1.7)$$

or

$$e = 1.60 \times 10^{-19} \text{ coulomb} \quad (1.8)$$

in SI units. It can be seen from the large negative power of 10 required in either case, that neither system of units is really appropriate for elementary particle, nuclei, atomic, or molecular physics (microscopic physics) where the electron charge is the relevant unit of charge. In Chapter 16, we will discuss other systems of units that are more appropriate for those cases.

The Coulomb's law force on each of the two charges is proportional to the product of the two charges, and each force is along their common axis. Thus Coulomb's law satisfies Newton's third law of equal and opposite forces. We could try to use the third law as a theoretical basis for the symmetrical appearance of the two charges, and the common action line of the forces, but Newton's third law is not a sturdy basis on which to build. Although it is satisfied in electrostatics, we will see in Chapter 7 that it is violated by the magnetic force between moving charges. A better principle is the conservation of linear and angular momentum. Conservation of linear momentum requires the two Coulomb forces to be equal and opposite. Conservation of angular momentum requires the two forces to be along the same action line. So we see that it is from these two conservation laws that the two forces in Coulomb's law are collinear, and the charges appear symmetrically.

The fact that the force is proportional to the first power of either charge is called **linearity**. A related, but logically somewhat more extended, assumption (verified by experiment) is that the Coulomb force due to two charges, located at different points, on a third is the vector sum of the two individual forces. This is called the **superposition principle** for the electric force. Extended to the force due to several charges, the superposition principle leads to the form

$$\mathbf{F} = \sum_n \frac{(\mathbf{r} - \mathbf{r}_n)qq_n}{|\mathbf{r} - \mathbf{r}_n|^3}, \quad (1.9)$$

for the force on a point charge q at \mathbf{r} due to other point charges q_n located at points \mathbf{r}_n . We can see from Eq. (1.9) that the \mathbf{r}/r^3 form of Coulomb's law is more convenient than using $\hat{\mathbf{r}}/r^2$, because the unit vector for $(\mathbf{r} - \mathbf{r}_n)$ would be awkward to use.

1.2 The Electric Field

At this point, it becomes useful to find the force in two stages by introducing the concept of the **electric field** \mathbf{E} , defined by

$$\mathbf{F} = q\mathbf{E} \quad (1.10)$$

for the force on a point charge q due to any collection of other charges. With this definition, Coulomb's law for the electric field due to a point charge q is

$$\mathbf{E} = \frac{q\hat{\mathbf{r}}}{r^2} \quad (1.11)$$

The electric field at a point \mathbf{r} due to a number of point charges q_n , located at positions \mathbf{r}_n , is given by

$$\mathbf{E}(\mathbf{r}) = \sum_n \frac{(\mathbf{r} - \mathbf{r}_n)q_n}{|\mathbf{r} - \mathbf{r}_n|^3}. \quad (1.12)$$

The effect of Eq. (1.9) can now be accomplished in two steps by first using Eq. (1.12) to find \mathbf{E} , and then Eq. (1.10) to give the force on the point charge q located at \mathbf{r} . Although introduced in this way as a mathematical convenience, we will see (as often happens in physics) that the electric field has important physical significance on its own, and is not merely a mathematical construct.

Equation (1.10) defines the electric field \mathbf{E} at the point \mathbf{r} in the presence of the charge q . However, care must be exercised if Eq. (1.10) is to be used to measure the electric field that existed at \mathbf{r} before the charge q was introduced. The introduction of the charge q can polarize any nearby matter, changing the field at point \mathbf{r} . This polarization can even produce an \mathbf{E} field where none existed before the introduction of charge q . (This is a common phenomenon in static electricity, causing lightning as well as other effects.) For this reason, the use of a test charge to measure a pre-existing electric field is accomplished by

$$\mathbf{E}_0 = \lim_{q \rightarrow 0} \frac{\mathbf{F}}{q}, \quad (1.13)$$

where \mathbf{E}_0 is the electric field that was present before the test charge was introduced.

We have thus far limited our considerations to point charges. In principle, this is all that is needed because it is believed that all charges appear as point charges of value $\pm e$ for leptons and $\pm\frac{2}{3}e$ or $\pm\frac{1}{3}e$ for quarks (the constituents of strongly interacting matter). However, the sum in Eq. (1.12) would have of the order of 10^{23} terms for macroscopic objects, and be impossible to use. For this reason, the concept of a continuous charge distribution as an abstraction of a huge number of point particles is introduced. That is, in Eq. (1.12), the sum on n is first taken over a large number of the point charges q_n that is still small compared with the total number of charges in a macroscopic sample. This leads to clusters of charge Δq_i each having n_i charges, so

$$\Delta q_i = \sum_{n=n_{i-1}}^{n_i} q_n. \quad (1.14)$$

The number of charges in each cluster can be large, and yet all charges in a cluster are still at about the same point, since the total number of charges is huge for a macroscopic sample. (For instance, one million atoms are contained in a cube 10^{-6} cm on a side.) This means that a very large number of point charges looks like, and can be well approximated by, a still large collection of effective point charges Δq_i . Then the electric field will be given by

$$\mathbf{E}(\mathbf{r}) = \sum_i \frac{(\mathbf{r} - \mathbf{r}_i)\Delta q_i}{|\mathbf{r} - \mathbf{r}_i|^3}. \quad (1.15)$$

In the limit that the number of charge clusters becomes infinite, (in this case, “infinity” is of the order of 10^{20}) and the net charge in each cluster approaches zero compared to the total charge, the sum approaches an integral over charge differentials dq' , and Eq. (1.15) is replaced by

$$\mathbf{E}(\mathbf{r}) = \int \frac{(\mathbf{r} - \mathbf{r}')dq'}{|\mathbf{r} - \mathbf{r}'|^3}. \quad (1.16)$$

In Eq. (1.16), there are two different position vectors, which we refer to as the source vector \mathbf{r}' and the field vector \mathbf{r} .

The form of the differential charge element dq' depends on the type of charge distribution. The integral operator $\int dq'$ becomes

$$\int dq' = \int \lambda(\mathbf{r}')dl' \text{ for a linear charge density } \lambda, \quad (1.17)$$

$$\int dq' = \int \sigma(\mathbf{r}')dA' \text{ for a surface charge density } \sigma, \quad (1.18)$$

$$\int dq' = \int \rho(\mathbf{r}')d\tau' \text{ for a volume charge density } \rho, \quad (1.19)$$

where dl' , dA' , and $d\tau'$ are differentials of length, area, and volume, respectively. Equation (1.16) can also be extended to point charges with the understanding that

$$\int dq' f(\mathbf{r}') = \sum_n q_n f(\mathbf{r}_n) \text{ for point charges } q_n \text{ at positions } \mathbf{r}_n. \quad (1.20)$$

The charge distribution on which \mathbf{E} acts can also be considered continuous, and then the force on the continuous charge distribution would be

$$F = \int dq \mathbf{E}(\mathbf{r}), \quad (1.21)$$

with $\int dq$ given as in Eqs. (1.17-1.20).

Equation (1.16) gives the static electric field for any charge distribution. The term “static,” as used here, means that all time derivatives of the charge

distribution are zero, or are neglected if the charges are moving. But using Eq. (1.16) is a “brute force” method that often requires complicated integration, and usually is not a practical way to find \mathbf{E} . (With the use of modern computers, it has become more practical to sometimes just do these integrals on the computer.) There are several simple geometries for which the use of symmetry simplifies the integrals, and some examples of these are given in the problems at the end of this chapter. Aside from these simple cases, better methods are usually needed to find the electric field.

1.3 Electric Potential

The work done by the electric field in moving a charge q from a point A to a point B along a path C is given by

$$W_C = q \int_{ABC} \mathbf{E} \cdot d\mathbf{r}, \quad (1.22)$$

where the notation \int_{ABC} means that the displacement $d\mathbf{r}$ is always along the path defined by the curve C from A to B . The definition of a conservative force field is one for which the net work done around any closed path is zero. We now show that this is true for the \mathbf{E} field of a point charge. This follows because the integrand of Eq. (1.22) can be written as the perfect differential $d(-1/r)$ when \mathbf{E} is given by Coulomb’s law,

$$d\left(\frac{-1}{r}\right) = \frac{d(r)}{r^2} = \frac{d(\mathbf{r} \cdot \mathbf{r})^{\frac{1}{2}}}{r^2} = \frac{\mathbf{r} \cdot d\mathbf{r}}{r^3}, \quad (1.23)$$

with the final form in Eq. (1.23) being the integrand for the net work on a unit charge using Coulomb’s law for \mathbf{E} . This result can be extended to the \mathbf{E} given by Eq. (1.15) or Eq. (1.16), because each of these are just linear sums of Coulomb’s law for a single point charge. Thus

$$\oint \mathbf{E} \cdot d\mathbf{r} = 0 \quad (1.24)$$

for any static electric field integrated around any closed path, and \mathbf{E} is said to be a **conservative field**. (The notation \oint indicates that the line integral is taken around a closed path.)

It follows that the work done on a unit charge by a conservative field in moving from point A to point B is independent of the path taken. This can be seen in Fig. 1.2 by picking any two points A and B on a closed path, and breaking the closed path integral into an integral from A to B along path C_1 , followed by an integral from B to A along path C_2 . Then

$$0 = \oint \mathbf{E} \cdot d\mathbf{r} = \int_{AB_1} \mathbf{E} \cdot d\mathbf{r} + \int_{BA_2} \mathbf{E} \cdot d\mathbf{r} = \int_{AB_1} \mathbf{E} \cdot d\mathbf{r} - \int_{AB_2} \mathbf{E} \cdot d\mathbf{r}, \quad (1.25)$$

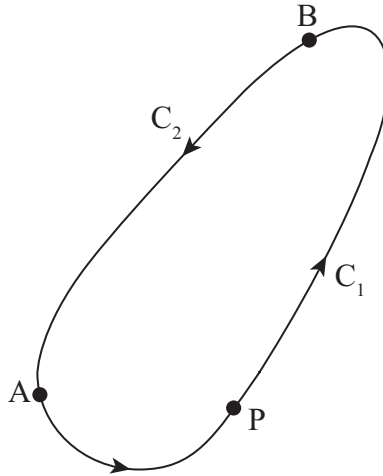


Figure 1.2: Closed integration path for $\oint \mathbf{E} \cdot d\mathbf{r}$.

so

$$\int_{AB_1} \mathbf{E} \cdot d\mathbf{r} = \int_{AB_2} \mathbf{E} \cdot d\mathbf{r}. \quad (1.26)$$

Since the $\int_{AB} \mathbf{E} \cdot d\mathbf{r}$ is independent of the path, it can be written as a scalar function ψ of only the endpoint positions.

$$\int_{AB} \mathbf{E} \cdot d\mathbf{r} = \psi(\mathbf{r}_A, \mathbf{r}_B) \quad (1.27)$$

Then, by the property of an integral that

$$\int_{ABC} = \int_{APC} + \int_{PBC} \quad (1.28)$$

for any point P on the path C, it follows that ψ must be the difference

$$\psi(\mathbf{r}_A, \mathbf{r}_B) = \phi(\mathbf{r}_A) - \phi(\mathbf{r}_B) \quad (1.29)$$

of a scalar function of position $\phi(\mathbf{r})$, evaluated at the two positions.

The scalar function $\phi(\mathbf{r})$ is defined as the **electric potential**, related to the electric field by

$$\phi(\mathbf{r}_B) - \phi(\mathbf{r}_A) = - \int_{\mathbf{r}_A}^{\mathbf{r}_B} \mathbf{E} \cdot d\mathbf{r}. \quad (1.30)$$

The integral defining the difference in potentials is independent of the path chosen from \mathbf{r}_A to \mathbf{r}_B . The potential ϕ defined in this way, with the minus sign before the integral in Eq. (1.30), has the physical significance of being the

potential energy per unit charge due to the electric field. Where practicable, it is convenient to take point A to be at infinity. Then

$$\phi(\mathbf{r}) = \int_{\mathbf{r}}^{\infty} \mathbf{E} \cdot d\mathbf{r}. \quad (1.31)$$

This integral represents the work done on a unit charge by the electric field in moving the charge from the point \mathbf{r} to infinity, which is equal to the work we would have to do to move the charge from infinity to \mathbf{r} .

For a point charge, the integration in Eq. (1.31) is easily done using Eq. (1.23). This results in Coulomb's law for the potential of a point charge:

$$\phi = \frac{q}{r}. \quad (1.32)$$

In the same way,

$$\phi(\mathbf{r}) = \sum_n \frac{q_n}{|\mathbf{r} - \mathbf{r}_n|} \quad (1.33)$$

for a collection of point charges, and

$$\phi(\mathbf{r}) = \int \frac{dq'}{|\mathbf{r} - \mathbf{r}'|} \quad (1.34)$$

for continuous charge distributions.

Since the electric potential is the potential energy per unit charge, its units in the esu system are ergs per esu of charge. A more convenient energy unit for microphysics (atomic, molecular, nuclear, or elementary particle) is the **electron volt** (eV), which is the potential energy that a particle with the electron charge ($e = 4.8 \times 10^{-10}$ esu) has in a potential of one volt (SI units). The esu statvolt equals 300 volts, so the esu result for the potential energy of a particle with charge e should be multiplied by 300 to give its energy in eV. For instance, the potential energy of an electron, a distance of 0.529 \AA from the proton in a hydrogen atom, is

$$U = -\frac{e^2}{r} = -300 \times \frac{4.8 \times 10^{-10}}{0.529 \times 10^{-8}} = -27.2 \text{ eV}. \quad (1.35)$$

Note that there is only one factor of e in the numerical calculation. The second e is included in the definition of the eV unit. The unit \AA (pronounced Angstrom) equals 10^{-8} cm, and is a convenient unit for atomic and molecular physics because the typical atomic size is about 1 \AA .

When using the electron volt energy unit, it is convenient to also give particle masses in energy units (anticipating relativity), even for low velocities. For instance, the velocity of an electron of kinetic energy $T=13.6$ eV can be given as

$$\frac{v}{c} = \left[\frac{2T}{mc^2} \right]^{\frac{1}{2}} = \left[\frac{27.2}{.511 \times 10^6} \right]^{\frac{1}{2}} = 7.3 \times 10^{-3}. \quad (1.36)$$

where we have used $mc^2 = 0.511$ MeV for the electron.

1.3.1 Potential gradient

It is usually easier to find the potential than it is to calculate the electric field by integrating over the charge distribution. What is needed then is a way to determine the electric field from the potential (i.e., the inverse process of Eq. (1.31)). We begin by first considering the relation between the potential ϕ and the field \mathbf{E} . A potential field is conveniently pictured by means of equipotentials, that is, surfaces along which ϕ is constant. A common example (in two dimensions) shown in Fig. 1.3 is a topographic map where the lines of equal altitude are equipotentials of the gravitational field. No work is done in moving along an equipotential, so the direction of \mathbf{E} (“line of force”) is everywhere perpendicular to the equipotential. Mathematically, that perpendicular direction is defined as the direction of the **gradient** of the potential. In our topographic example, this direction is the steepest direction up the hill. Experimentally, this would be opposite the direction a ball would roll if placed at rest on the hillside.

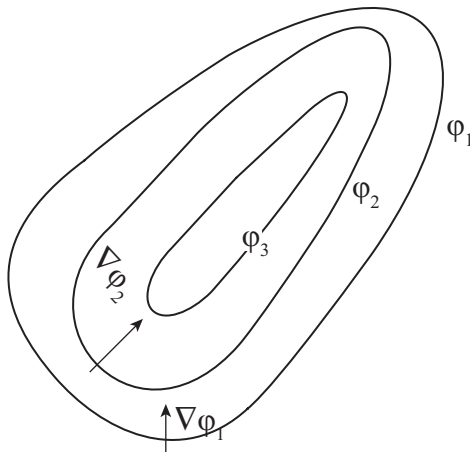


Figure 1.3: Equipotentials ϕ_i and gradients $\nabla\phi_i$.

The magnitude of the gradient is defined to be the rate of change of the potential with respect to distance in the direction of maximum increase. For infinitesimal displacements, an equipotential surface in three dimensions can be approximated by its tangent plane so the change in a scalar field in an infinitesimal displacement $d\mathbf{r}$ will vary as the cosine of the angle between the direction of maximum gradient and $d\mathbf{r}$. Then a vector gradient (written as **grad** ϕ) can be defined by

$$d\phi(\mathbf{r}) = d\mathbf{r} \cdot \mathbf{grad}\phi. \quad (1.37)$$

From Eq. (1.37), it can be seen that **grad** ϕ determines how much the scalar field ϕ will change when you move a short distance $d\mathbf{r}$. The definition of **grad** ϕ

by Eq. (1.37) may seem a bit indirect, but Eq. (1.37) can be used to give the direct definition that

$$\mathbf{grad}\phi = \hat{\mathbf{n}} \frac{d\phi}{|\mathbf{dr}|}. \quad (1.38)$$

In Eq. (1.38), the unit vector $\hat{\mathbf{n}}$ is in the direction of maximum increase of ϕ , and \mathbf{dr} is taken in that direction of maximum increase. The direction of the gradient will always be in the direction of maximum increase and perpendicular to the equipotentials of the scalar function.

The rate of change of the scalar field in a general direction $\hat{\mathbf{a}}$, not necessarily the direction of maximum change, can be defined by choosing \mathbf{dr} in that direction and dividing both sides of Eq. (1.37) by the magnitude $|\mathbf{dr}|$. This is called the **directional derivative** of ϕ , defined by $\hat{\mathbf{a}} \cdot \mathbf{grad}\phi$ for the rate of change of ϕ in the direction $\hat{\mathbf{a}}$.

To get a slightly different feel for the gradient of a scalar field, we consider its application in a specific coordinate system. (Up to now, we have used no specific coordinate system. For the most part, we will continue that practice since it permits more generality and introduces less algebraic complexity.) In Cartesian (x, y, z) coordinates, an infinitesimal displacement is given by

$$\mathbf{dr} = \hat{\mathbf{i}}dx + \hat{\mathbf{j}}dy + \hat{\mathbf{k}}dz. \quad (1.39)$$

Then, Eq. (1.37) defining the gradient can be written

$$\begin{aligned} d\phi(x, y, z) &= (\hat{\mathbf{i}}dx + \hat{\mathbf{j}}dy + \hat{\mathbf{k}}dz) \cdot \mathbf{grad}\phi \\ &= (\mathbf{grad}\phi)_x dx + (\mathbf{grad}\phi)_y dy + (\mathbf{grad}\phi)_z dz. \end{aligned} \quad (1.40)$$

At the same time, the differential of the function $\phi(x, y, z)$ of three variables is given by

$$d\phi(x, y, z) = \partial_x \phi dx + \partial_y \phi dy + \partial_z \phi dz. \quad (1.41)$$

Note that we will use the notation ∂_x rather than the more cumbersome $\frac{\partial}{\partial x}$ to represent the partial derivative in the x direction.

Comparing Eqs. (1.40) and (1.41) for the same differential, and using the fact that the displacements dx, dy, dz are independent and arbitrary, we see that

$$\mathbf{grad}\phi = \hat{\mathbf{i}}\partial_x \phi + \hat{\mathbf{j}}\partial_y \phi + \hat{\mathbf{k}}\partial_z \phi. \quad (1.42)$$

Equation (1.42) holds only in Cartesian coordinates. The form of $\mathbf{grad}\phi$ is somewhat more complicated in other coordinate systems.

Equation (1.42) can be written in terms of a vector differential operator ∇ , given in Cartesian coordinates by

$$\nabla = \hat{\mathbf{i}}\partial_x + \hat{\mathbf{j}}\partial_y + \hat{\mathbf{k}}\partial_z. \quad (1.43)$$

Then, the gradient of a scalar field can be written as

$$\mathbf{grad}\phi = \nabla\phi. \quad (1.44)$$

The representation of **grad** by the vector differential operator ∇ (usually called **del**) is not limited to Cartesian coordinates, although its form is not given by Eq. (1.43) for other coordinate systems. We will derive a coordinate independent definition of ∇ later.

From Eq. (1.30), relating ϕ to \mathbf{E} , we see that the integrand in Eq. (1.30) is the differential

$$d\phi = -\mathbf{E}\cdot d\mathbf{r}. \quad (1.45)$$

This is just the definition of the gradient of ϕ , so comparing Eq. (1.45) with Eq. (1.37), we see that

$$\mathbf{E} = -\nabla\phi. \quad (1.46)$$

This permits us to find \mathbf{E} once ϕ has been determined. In words, Eq. (1.46) states that the electric field is the negative gradient of the potential, or, more simply, “E equals minus Del phi.”

The actual calculation of $\nabla\phi$ can be made using a coordinate system, but it is usually better to use the definition of the gradient in Eq. (1.38) to find it for various functions of the position vector \mathbf{r} directly. We start with r , the magnitude of \mathbf{r} , treated as a scalar field. Its maximum rate of change is in the $\hat{\mathbf{r}}$ direction, and its derivative in that direction is $dr/dr = 1$. So

$$\nabla r = \hat{\mathbf{r}}. \quad (1.47)$$

Next, we consider any scalar function, $f(r)$, of the magnitude of \mathbf{r} . The direction of maximum rate of change of $f(r)$ will also be $\hat{\mathbf{r}}$, and its derivative in that direction is df/dr . So

$$\nabla f(r) = \hat{\mathbf{r}} \frac{df}{dr}. \quad (1.48)$$

Applying this result to Coulomb’s law for the potential due to a point charge gives

$$\mathbf{E} = -\nabla\left(\frac{q}{r}\right) = -q\hat{\mathbf{r}}\frac{d}{dr}\left(\frac{1}{r}\right) = q\frac{\hat{\mathbf{r}}}{r^2}, \quad (1.49)$$

so we have derived Coulomb’s law for \mathbf{E} from Coulomb’s law for ϕ .

The same derivation works for the electric field of a collection of point charges or a continuous charge distribution. For a continuous distribution, this becomes

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= -\nabla \int \frac{dq'}{|\mathbf{r} - \mathbf{r}'|} \\ &= -\int dq' \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \\ &= \int \frac{(\mathbf{r} - \mathbf{r}')dq'}{|\mathbf{r} - \mathbf{r}'|^3}, \end{aligned} \quad (1.50)$$

where the ∇ can be taken under the integral because it is a partial derivative that does not act on \mathbf{r}' . We have also used the fact that $(\mathbf{r} - \mathbf{r}')$ is just \mathbf{r} with the origin displaced by the constant vector \mathbf{r}' , so

$$\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -\frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}. \quad (1.51)$$

A useful vector identity involving the gradient follows from applying the definition of the gradient to $\nabla(\mathbf{r} \cdot \mathbf{A})$, with \mathbf{A} being a constant vector:

$$d\mathbf{r} \cdot \nabla(\mathbf{r} \cdot \mathbf{A}) = d(\mathbf{r} \cdot \mathbf{A}) = d\mathbf{r} \cdot \mathbf{A}. \quad (1.52)$$

Since $d\mathbf{r}$ is an arbitrary displacement, we get the useful result

$$\nabla(\mathbf{r} \cdot \mathbf{A}) = \mathbf{A}. \quad (1.53)$$

1.4 Gauss's Law

A method for using symmetry to find \mathbf{E} without integrating over the charge distribution is given by **Gauss's law**. We first derive Gauss's law from Coulomb's law for a single point charge. We start with the surface integral

$$\oint d\mathbf{A} \cdot \mathbf{E} = q \oint d\mathbf{A} \cdot \frac{\mathbf{r}}{r^3} \quad (1.54)$$

of the normal component of \mathbf{E} over a closed surface surrounding the point charge, as shown in Fig. 1.4.

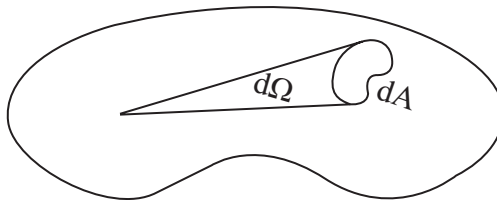


Figure 1.4: The solid angle $d\Omega$ subtended by the surface differential $d\mathbf{A}$ for Gauss's law.

The vector differential of area $d\mathbf{A}$ is an infinitesimal surface element of magnitude dA . Since it is infinitesimal, it approaches a plane surface, tangent to the closed surface. By convention, its vector direction is along the outward normal to the closed surface. The integrand of this surface integral can be recognized as the definition of the solid angle subtended by the differential surface element $d\mathbf{A}$, as can be seen on Fig. 1.4.

$$d\Omega = \frac{\hat{\mathbf{r}} \cdot d\mathbf{A}}{r^2} \quad (1.55)$$

Then the surface integral can be written as

$$\oint \mathbf{dA} \cdot \mathbf{E} = q \oint d\Omega = 4\pi q, \quad (1.56)$$

with the factor 4π arising as the magnitude of the total solid angle of any closed surface.

If the point charge q were located outside the closed surface, then the surface integral would be zero. This can be seen in Fig. 1.5. If a plane surface is made to cut the closed integration surface into two parts, then the integrated solid angle over each part of the surface will equal in magnitude the solid angle subtended by the part of the plane surface inside the closed surface. But the two solid angles will be of opposite sign and just cancel. Thus the integral over the closed surface will be $4\pi q$ for a point charge inside the surface, and zero for a point charge outside the surface.

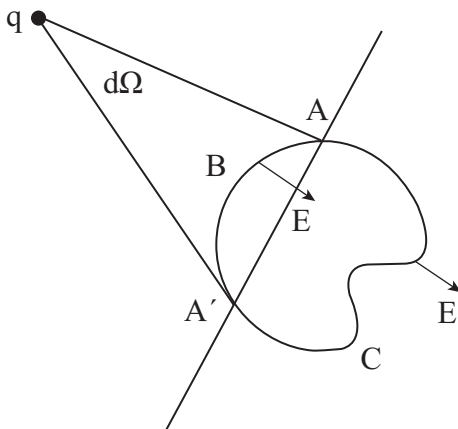


Figure 1.5: Gauss's law for charge outside a closed surface. The solid angle $d\Omega$ subtended by each of the curved surfaces ABA' and ACA' are equal to the solid angle subtended by the plane AA' .

For a collection of point charges, only those inside the surface will contribute to the integral, and we have Gauss's law

$$\oint \mathbf{dA} \cdot \mathbf{E} = 4\pi Q_{\text{enclosed}}, \quad (1.57)$$

where Q_{enclosed} is the net charge within the surface. Since a continuous charge distribution is a very large number of point charges, Gauss's law for a continuous distribution is also given by Eq. (1.57), with

$$Q_{\text{enclosed}} = \int \rho(\mathbf{r}) d\tau, \quad (1.58)$$

where the integral is over the volume enclosed by the closed surface. For SI units, Gauss's law is written

$$\oint \mathbf{dA} \cdot \mathbf{E} = \frac{1}{\epsilon_0} Q_{\text{enclosed}}, \quad \mathbf{SI} \quad (1.59)$$

without the 4π , but introducing ϵ_0 . Actually, the 4π in Eq. (1.57), the Gaussian form of Gauss's law, is a good reminder that it comes from the integral over the solid angle.

Gauss's law provides a powerful and simple method to find \mathbf{E} whenever there is enough symmetry to enable the surface integral to be done without integration. But, if use of Gauss's law requires a complicated surface integration, then another method should be used to find \mathbf{E} .

1.4.1 Examples of Gauss's law

Point charge

As a simple example of Gauss's law, we use it to derive Coulomb's law for a point charge, demonstrating the steps used in application of Gauss's law. The first step is to recognize the symmetry of the charge configuration, which, in the case of an isolated point charge, is spherical symmetry about the point charge. The type of symmetry dictates the **Gaussian surface** to be used for the surface integral. For the point charge, this is a sphere, of any radius r , centered at the point charge so as to make use of the symmetry, as shown in Fig. 1.6. Note that the Gaussian surface is just a mathematical surface that need not be (and usually isn't) any physical surface of the problem.

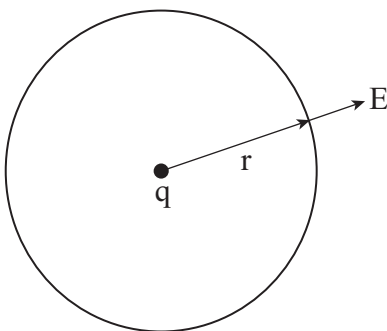


Figure 1.6: Gaussian sphere for a single point charge.

Next the symmetry is used to make simplifying observations about the \mathbf{E} field at the Gaussian surface. For the point charge, we first observe that \mathbf{E} must be in the radial direction with respect to the charge. This follows, most simply,

from the principle of insufficient reason. That is, looking at Fig. 1.6, there is no more reason for \mathbf{E} to be directed to the right of radial than to the left, since there is no reference point other than the charge to define left or right. Another, more mathematical, derivation of the radial direction of \mathbf{E} is the fact that only one vector, the position vector \mathbf{r} , can be defined for this geometry. Thus the vector function \mathbf{E} must be given only in terms of this vector and can only be $\mathbf{E}(\mathbf{r}) = E(\mathbf{r})\hat{\mathbf{r}}$. (This reasoning will be used later in the book to simplify other vector integrations.)

The next observation is that the radial \mathbf{E} must have the same magnitude for all points on the sphere because all such points are equivalent, given the spherical symmetry. These two observations (that \mathbf{E} is radial and constant in magnitude around the spherical surface) lead to the simplification of Gauss's law for this case that

$$\oint \mathbf{E} \cdot d\mathbf{A} = E \oint dA = 4\pi r^2 E = 4\pi q. \quad (1.60)$$

The $\oint dA$ in Eq. (1.60) was done by just observing that it is the surface area of the sphere. Then dividing by $4\pi r^2$, gives

$$\mathbf{E}(\mathbf{r}) = \frac{q\hat{\mathbf{r}}}{r^2}, \quad (1.61)$$

which is just Coulomb's law for the electric field.

Because Gauss's law is derived from Coulomb's law, and Coulomb's law for the electric field can be derived from Gauss's law, the two laws are mathematically equivalent. Either one could be chosen as the starting point for electrostatics. Historically, Coulomb's law was discovered first, but Gauss's law is more general in the sense that Coulomb's law is just one of the simple applications of Gauss's law. Also, as we will see later in discussing Faraday's "ice bucket" experiment, Gauss's law can be verified to greater accuracy than Coulomb's law.

However, Gauss's law by itself does not lead to the Coulomb law of force, Eq.(1.1). Starting from Gauss's law, the additional assumption that the charge appearing in $\mathbf{F} = q\mathbf{E}$ is the same (i.e., interchangeable) with the charge appearing in Gauss's law. This was implicitly assumed in Coulomb's force law, leading to Newton's third law for the Coulomb force. The equivalence of these two charges, the **source charge** in Gauss's law, and the **force charge** in $\mathbf{F} = q\mathbf{E}$, must be an additional assumption in classical Electromagnetism.

1.4.2 Spherically symmetric charge (and mass) distributions

You may have noticed in the derivation of Coulomb's law from Gauss's law that the fact that the charge was a point charge never entered. The only property

needed for the charge distribution was that it was spherically symmetric and entirely contained within the Gaussian sphere. Thus the derivation would work just as well for any such charge distribution, and we have the result that the electric field outside any spherically symmetric charge distribution is the same as that for a point charge of the same net charge, located at the center of spherical symmetry. This means, for instance, that a uniformly charged sphere, or a uniformly charged shell (or collection of shells), would have the same electric field (and, therefore, the same electric potential) beyond the charge distribution as a point charge, and be indistinguishable from a point charge. Of course, inside the charge distribution, the electric field would be modified, and not fall off like $1/r^2$.

This result would have been very important in the development of gravitational theory (which is mathematically equivalent to electrostatics), had Newton known Gauss's law. Newton formulated his law of gravitation for point masses, just as the later Coulomb's law was formulated for point charges. However, Newton knew that the Earth and the Moon were not point masses. How could the point mass formula work so well for extended masses? Newton predated Gauss so he could be excused for not knowing Gauss's law. The only formulation he had for the force on an extended object B exerted by an extended object A was (we use Coulomb's law, but the arguments would be the same for Newton's law of gravity).

$$\begin{aligned} \mathbf{F}_B &= \int \rho_B(\mathbf{r}_B) \mathbf{E}_A(\mathbf{r}_B) d\tau_B \\ &= \int d\tau_B \int d\tau_A \rho_B(\mathbf{r}_B) \rho_A(\mathbf{r}_A) \frac{(\mathbf{r}_B - \mathbf{r}_A)}{|\mathbf{r}_B - \mathbf{r}_A|^3}. \end{aligned} \quad (1.62)$$

The double integral in Eq. (1.62) satisfies Newton's third law nicely, but is horrendous to integrate directly, even for two uniformly charged spheres. The double integral does not look anything like the force between two point charges (point masses for Newton). Poor Newton worked for many years. He actually used a complicated geometrical argument (given in his Principia) to show that the force between two spheres, each with spherically symmetric density, was exactly the same as the force between two point masses. So all his relatively simple equations for point masses also worked for large physical objects, as long as they were spherically symmetric.

We now show how Newton could have saved a lot of work had he known Gauss's law. Consider the force between the two extended, but spherically symmetric and non-overlapping, charge distributions, A and B, shown in Fig. 1.7a. The electric field due to A would appear at sphere B like that of a point charge, by the Gauss's law argument given above. So, the $\mathbf{E}_A(\mathbf{r}_B)$ in Eq.(1.62) would be that of a point charge, but this would still leave an integral over the extended

object B. However, we now know that the force on B is the same as if the extended object A were a point charge, as shown in Fig. 1.7b.

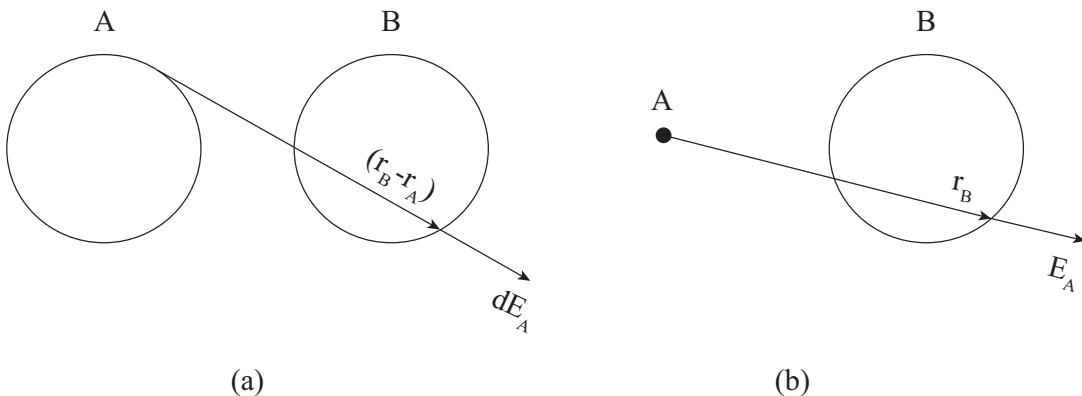


Figure 1.7: Gauss's law for two uniformly charged spheres. By Gauss's law, the sphere A in Fig. (a) can be replaced by a point charge in Fig. (b).

We next use Newton's third law to observe that the force on the (equivalent) point charge A due to the extended charge B is the same as that on B due to A. In calculating the force on the point charge A, we use Gauss's law again to effectively replace the extended object B by a point charge, so the force on A is the same as that due to a point charge B. We thus see that the force between A and B is the same as that between two point charges, without performing any integral.

Line charge

Gauss's law can be applied to other simple symmetries. For an infinitely long (in practical terms, "infinitely long" means length $\gg r_{\perp}$, and " \gg " will usually mean about a factor of 10, although \gg is often sneaked in for lower ratios). straight line charge with linear charge density λ , the Gaussian surface is a cylinder of arbitrary length L and radius r_{\perp} , coaxial with the line charge, as shown in Fig. 1.8.

From the axial symmetry, \mathbf{E} is everywhere directed straight out from the line charge, and can depend only on the perpendicular distance from the line charge, r_{\perp} . The integral for Gauss's law is over three surfaces, the curved surface of the cylinder (I), and the two end caps (II and III). The integrals over the endcaps vanish because \mathbf{E} is always parallel to the surface of the endcaps. E_{\perp} is constant

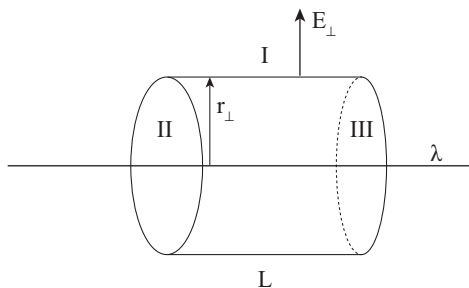


Figure 1.8: Gaussian surface for a line charge.

on the curved surface, and can be taken out of the integral, which becomes just the curved surface area. Then, Gauss's law becomes

$$\oint \mathbf{E} \cdot d\mathbf{A} = E_{\perp} \int_I dA = 2\pi r_{\perp} L E_{\perp} = 4\pi \lambda L, \quad (1.63)$$

so

$$E_{\perp} = \frac{2\lambda}{r_{\perp}}. \quad (1.64)$$

The potential for the line charge is the radial integral of E_{\perp}

$$\phi(\mathbf{r}) = -2\lambda \int_r^{r_0} \frac{dr}{r} = 2\lambda \ln(r/r_0). \quad (1.65)$$

E_{\perp} falls off too slowly for the integral for ϕ to converge at infinity, and an infinite amount of work would have to be done to bring a charge from infinity to finite distances for the infinite line charge. Therefore, an arbitrary finite point, r_0 , has been chosen for us to set $\phi(r_0) = 0$. As with spherical symmetry, the Gauss's law derivation for the wire would also hold outside any axially symmetric charge distribution that is uniform along its axis. So, \mathbf{E} outside any such charge distribution is the same as that for a uniform line charge.

Infinite plane

We next look at an infinite plane sheet with a constant surface charge density σ . The symmetry of the infinite plane sheet requires that \mathbf{E} be perpendicular to the plane of the sheet, and not depend on position parallel to the sheet. The appropriate Gaussian surface, shown in Fig. 1.9, is a **Gaussian pillbox**, a flat box with identical parallel ends (I and II) of arbitrary area A and any shape. (Gauss meant a flat box used to carry healing pills, and not the deadly military fort of the same general shape.) The pillbox is oriented parallel to the sheet of

charge and is bisected by the sheet. \mathbf{E} is parallel to the side (III) of the pillbox, so the only contribution to the Gauss's law integral is from the two flat surfaces.

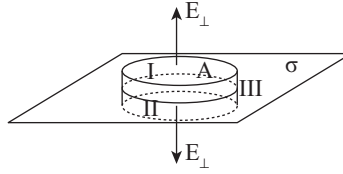


Figure 1.9: Gaussian pillbox for an infinite plane.

From the symmetry of the sheet, \mathbf{E} has the same magnitude, but opposite direction, on either side of the sheet. Gauss's law then leads to

$$\oint \mathbf{E} \cdot d\mathbf{A} = 2E_{\perp} \int_I dA = 2E_{\perp} A = 4\pi\sigma A, \quad (1.66)$$

so

$$E_{\perp} = 2\pi\sigma, \quad (1.67)$$

for an infinite sheet of charge with constant surface charge density σ . Note that \mathbf{E} is independent of the distance from the infinite plane sheet, as long as the sheet still looks infinite.

The above derivation can be modified slightly to give the discontinuity in \mathbf{E} , when a charged surface is crossed. For the discontinuity, we consider infinitesimal distances on either side of the surface so that any continuous surface looks like a plane. Then, using the Gaussian pillbox in Fig. 1.9, the discontinuity in \mathbf{E} is given by

$$\hat{\mathbf{n}} \cdot \mathbf{E}_1 - \hat{\mathbf{n}} \cdot \mathbf{E}_2 = 4\pi\sigma, \quad (1.68)$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the plane, pointing from region 2 to region 1. Equation (1.67) is a special case of this result when the only source of \mathbf{E} is the surface charge.

1.5 The Variation of \mathbf{E}

We have seen that the variation of a scalar field ϕ is determined by its gradient, so $d\phi = \mathbf{dr} \cdot \nabla\phi$. A vector field can have two different types of variation. It can vary along its direction, for example, like the flow velocity field, \mathbf{v} , of a stream as the slope gets steeper. The vector field can also vary across its direction, as

when the velocity is faster in the middle of the stream than near the edges. How can these two variations be measured?

1.5.1 Divergence

We now give a physical definition of what is called the **divergence** of a vector field. The increase of a vector field along its direction is shown in Fig. 1.10. A measure of the strength of the field is the density of **lines of force** in the figure, with the increase in the field indicated by increasing lines of force. We construct a mathematical volume V enclosed by a surface S , as shown in the figure. The increase in \mathbf{E} can be seen in the figure as more lines of \mathbf{E} leaving the volume than entering it.

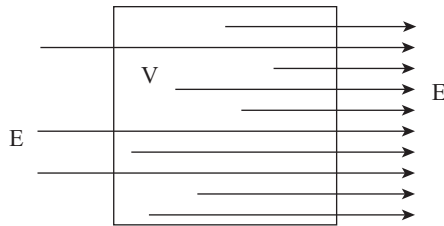


Figure 1.10: Divergence of lines of \mathbf{E} . More lines leave the volume than enter it.

A quantitative measure of the excess of lines leaving the volume is given by the integral $\oint_S \mathbf{E} \cdot d\mathbf{A}$. This integral can be used to define an average divergence (written as “div”) of the lines of the vector field. That is

$$\langle \text{div } \mathbf{E} \rangle_V = \frac{1}{V} \oint_S \mathbf{E} \cdot d\mathbf{A}, \quad (1.69)$$

where the notation $\langle \text{div } \mathbf{E} \rangle_V$ denotes the average of $\text{div } \mathbf{E}$ over the volume V . The value of $\text{div } \mathbf{E}$ at a point can be defined by shrinking the integral about the point, so that

$$\text{div } \mathbf{E} = \lim_{V \rightarrow 0} \frac{1}{V} \oint_S \mathbf{E} \cdot d\mathbf{A} \quad (1.70)$$

gives the divergence of the vector field at a point (if the limit exists), and is a measure of its rate of increase along the direction of the vector field.

As we did with the gradient, we now show what the divergence would look like in Cartesian coordinates. Figure 1.11 shows an infinitesimal volume (a parallelepiped in Cartesian coordinates) of dimensions $\Delta x \times \Delta y \times \Delta z$, which will

shrink to zero at the point x, y, z . The surface integral in the definition of $\text{div}\mathbf{E}$ is over the six faces of the parallelepiped, I-VI, so the integral can be written as

$$\text{div } \mathbf{E} = I + II + III + IV + V + VI, \quad (1.71)$$

where I indicates the integral over face I, and similarly for the other faces.

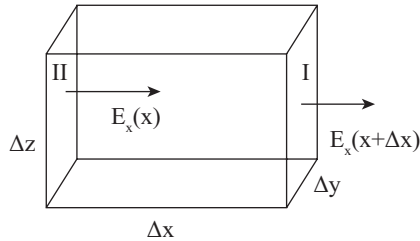


Figure 1.11: Volume element in Cartesian coordinates.

We concentrate first on faces I and II, each parallel to the $y - z$ plane. In the limit as both Δy and Δz approach zero, the integral over face I approaches $E_x(x + \Delta x, y, z)\Delta y\Delta z$, and that over face II approaches $-E_x(x, y, z)\Delta y\Delta z$, provided that E_x is continuous at the point x, y, z . So, for these two faces

$$\begin{aligned} I + II &= \lim_{\Delta x, \Delta y, \Delta z \rightarrow 0} \frac{[E_x(x + \Delta x, y, z)\Delta y\Delta z - E_x(x, y, z)\Delta y\Delta z]}{\Delta x\Delta y\Delta z} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[E_x(x + \Delta x, y, z) - E_x(x, y, z)]}{\Delta x} \\ &= \partial_x E_x, \end{aligned} \quad (1.72)$$

where the last step follows from the definition of the partial derivative.

The integrals over the other four faces are done in the same way, leading to similar results, with the substitutions $x \rightarrow y$ and then $x \rightarrow z$, so

$$\text{div } \mathbf{E} = \partial_x E_x + \partial_y E_y + \partial_z E_z \quad (1.73)$$

in Cartesian coordinates. The form of Eq. (1.73) suggests that $\text{div } \mathbf{E}$ could be written as a dot product

$$\text{div } \mathbf{E} = \nabla \cdot \mathbf{E}, \quad (1.74)$$

of the vector differential operator ∇ with \mathbf{E} . Equation (1.74) can be stated in words as “divergence \mathbf{E} equals **Del dot E**.” The representation of div by the vector differential operator $\nabla \cdot$ is not limited to Cartesian coordinates, although

its explicit form is not given by Eq. (1.73) for other coordinate systems. We will derive a coordinate independent definition of ∇ shortly.

Equation (1.69) defines the average of the divergence over a finite volume. Using the definition of a volume average that

$$\langle \nabla \cdot \mathbf{E} \rangle_V = \frac{1}{V} \int_V \nabla \cdot \mathbf{E} d\tau, \quad (1.75)$$

Eq. (1.69) can be rewritten as

$$\int_V \nabla \cdot \mathbf{E} d\tau = \oint_S \mathbf{dA} \cdot \mathbf{E}. \quad (1.76)$$

In this form, it is called the **divergence theorem**. Our derivation of the divergence theorem has been so simple because we have effectively defined the average divergence for a finite volume by the divergence theorem, and then shown that this average divergence approaches other definitions of the divergence at a point as the volume shrinks to the point.

The definition of the divergence given by Eq. (1.70) can be used to evaluate the divergence of the position vector. The definition gives

$$\begin{aligned} \nabla \cdot \mathbf{r} &= \lim_{V \rightarrow 0} \frac{1}{V} \oint \mathbf{r} \cdot \mathbf{dA} \\ &= \lim_{V \rightarrow 0} \frac{1}{V} \oint R^3 d\Omega = 3, \end{aligned} \quad (1.77)$$

where we have used the fact that

$$V = \oint d\Omega \int_0^R r^2 dr = \frac{1}{3} \oint R^3 d\Omega. \quad (1.78)$$

(Note that the R in the integral $\oint R^3 d\Omega$ refers to the distance from the origin to the bounding surface, and can be a function of angle.)

We next calculate the divergence of the electric field in Coulomb's law to get

$$\begin{aligned} \nabla \cdot \left[\frac{q\mathbf{r}}{r^3} \right] &= q \frac{\nabla \cdot \mathbf{r}}{r^3} + q\mathbf{r} \cdot \nabla \frac{1}{r^3} \\ &= \frac{3q}{r^3} - \frac{3q\mathbf{r} \cdot \hat{\mathbf{r}}}{r^4} = 0, \quad r \neq 0. \end{aligned} \quad (1.79)$$

The restriction $r \neq 0$ is necessary because both terms in Eq. (1.79) are singular at $r = 0$.

In Eq. (1.79) we have demonstrated the procedure for applying the vector differential operator ∇ to a combination of functions. We have made use of the fact that the operator ∇ has two distinct properties:

- 1. ∇ is a differential operator.**

2. ∇ is a vector.

Because ∇ is a differential operator, it acts on functions one at a time, just as in $d(uv) = u dv + v du$. We also follow the convention that the differential operator acts only on functions to its right, so the order in which ∇ appears must be to the left of the functions it acts on and to the right of the other functions. As a vector, ∇ must behave in any expansion like any other vector.

The use of both of these properties can be seen in Eq. (1.79). There are two terms because ∇ acts separately on the \mathbf{r} and on the $1/r^3$, and, in each term, the ∇ remains dotted with the \mathbf{r} , and to the left of the function it acts on. In every case, strict adherence to these two properties of ∇ will lead to the correct evaluation of vector derivatives.

1.5.2 Dirac delta function

We now investigate the behavior of $\nabla \cdot [\mathbf{r}/r^3]$ at $r = 0$. In fact, something quite dramatic happens at the origin to the divergence of \mathbf{r}/r^3 , as can be seen by applying the divergence theorem to \mathbf{r}/r^3 :

$$\int d\tau \nabla \cdot \left[\frac{\mathbf{r}}{r^3} \right] = \oint \frac{\mathbf{dA} \cdot \mathbf{r}}{r^3} = \oint d\Omega = 4\pi. \quad (1.80)$$

So, even though $\nabla \cdot [\mathbf{r}/r^3]$ vanishes at all but one point, its volume integral is not zero. This property is consistent with one definition of the **Dirac delta function** in three dimensions:

$$\begin{aligned} \int_V d\tau \delta(\mathbf{r}) &= 1, & \text{if } \mathbf{r} = 0 \text{ inside } V, \\ \int_V d\tau \delta(\mathbf{r}) &= 0, & \text{if } \mathbf{r} \neq 0 \text{ inside } V. \end{aligned} \quad (1.81)$$

Then

$$\nabla \cdot \left[\frac{\mathbf{r}}{r^3} \right] = 4\pi \delta(\mathbf{r}). \quad (1.82)$$

We can now take the divergence of the Coulomb electric field, including the origin, to get

$$\nabla \cdot \left[\frac{q\mathbf{r}}{r^3} \right] = 4\pi q \delta(\mathbf{r}). \quad (1.83)$$

To apply the divergence theorem to integrals over continuous charge distributions, we extend Eq. (1.82) to

$$\nabla \cdot \left[\frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right] = 4\pi \delta(\mathbf{r} - \mathbf{r}'), \quad (1.84)$$

where the constant vector \mathbf{r}' just shifts the origin of coordinates from $\mathbf{0}$ to \mathbf{r}' . In any integral over a volume V containing the point $\mathbf{r}' = \mathbf{r}$, the region of

integration can be shrunk to an infinitesimal volume surrounding \mathbf{r} . Then the integral including $\delta(\mathbf{r} - \mathbf{r}')$ has the property

$$\begin{aligned}\int_V \delta(\mathbf{r} - \mathbf{r}') f(\mathbf{r}') d\tau' &= f(\mathbf{r}), & \text{if } \mathbf{r}' = \mathbf{r} \text{ inside } V \\ \int_V \delta(\mathbf{r} - \mathbf{r}') f(\mathbf{r}') d\tau' &= 0, & \text{if } \mathbf{r}' \neq \mathbf{r} \text{ inside } V,\end{aligned}\quad (1.85)$$

provided that $\lim_{\mathbf{r}' \rightarrow \mathbf{r}} f(\mathbf{r}')$ exists. So, an integral with a delta function is the simplest integral to do. The integration just involves evaluating the rest of the integrand at the point where the argument of the delta function vanishes. Equation (1.85) is a somewhat better definition of the Dirac delta function than Eq. (1.81) because it permits a more general definition of the delta function with respect to a class of functions $f(\mathbf{r})$. Then, Eq. (1.81) follows from this definition if $f(\mathbf{r}')$ is chosen to be 1.

We must emphasize that the Dirac delta function is not a mathematical function in the strict sense. In fact, as a function, it does not make sense. It would vanish everywhere, except where it was not defined (loosely speaking, “infinite”). That is why we have been careful, in either definition, to define the delta function only in terms of its property in integrals. When we write it in equations where we do not integrate, such as Eq. (1.84), it is always with the understanding that the delta function will only be given physical meaning in a subsequent integration. That is, in equations like Eq. (1.84), the delta function is just an indication of how to perform a pending integration.

Applying the divergence to the Coulomb integral for the electric field of a volume distribution of charge leads to

$$\begin{aligned}\nabla \cdot \mathbf{E}(\mathbf{r}) &= \nabla \cdot \int \frac{(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}') d\tau'}{|\mathbf{r} - \mathbf{r}'|^3} \\ &= \int \rho(\mathbf{r}') d\tau' \nabla \cdot \left[\frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right] \\ &= 4\pi \int \rho(\mathbf{r}') d\tau' \delta(\mathbf{r} - \mathbf{r}').\end{aligned}\quad (1.86)$$

Doing the delta function integral gives

$$\nabla \cdot \mathbf{E} = 4\pi\rho \quad (1.87)$$

for any continuous charge distribution.

Equation (1.87) has been derived starting from Coulomb’s law, which was the historical order of development. However, the theory of electrostatics could also start with the partial differential equation given by Eq. (1.87). Then, Gauss’s law can be derived by applying the divergence theorem

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \int_V \nabla \cdot \mathbf{E} d\tau = 4\pi \int_V \rho(\mathbf{r}) d\tau = 4\pi Q_{\text{enclosed}}. \quad (1.88)$$

And, as we have shown, Gauss's law can be used to derive Coulomb's law for a point charge. So we see there are three, mathematically equivalent "starting points" for electrostatics (Coulomb's law, Gauss's law, $\nabla \cdot \mathbf{E} = 4\pi\rho$).

Since Gauss's law can be derived from $\nabla \cdot \mathbf{E} = 4\pi\rho$, and $\nabla \cdot \mathbf{E} = 4\pi\rho$ can be derived from Gauss's law (by following the steps of Eq. (1.87) backward), they are mathematically equivalent. For this reason, the equation $\nabla \cdot \mathbf{E} = 4\pi\rho$ is sometimes called "the differential form of Gauss's law." However, the two laws represent quite different physical manifestations.

Equation (1.87) can be put in terms of the electric potential ϕ , leading to **Poisson's equation**

$$\nabla \cdot (\nabla \phi) = \nabla^2 \phi = -4\pi\rho. \quad (1.89)$$

Equation (1.89) introduces the **Laplacian** differential operator ∇^2 defined by the application of the divergence to the gradient of a scalar. In Cartesian coordinates, the Laplacian operator is given by

$$\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2, \quad (1.90)$$

but it is more complicated in other coordinate systems. The homogeneous form of Poisson's equation, with the source function $\rho = 0$,

$$\nabla^2 \phi = 0 \quad (1.91)$$

is called **Laplace's equation**.

1.5.3 Curl

Next, we look at how \mathbf{E} can vary across its direction, and we give a physical definition of the **curl** of a vector field. Figure 1.12 shows a vector field having such a variation, with the density of lines being proportional to the strength of the field. If this were a velocity field, such as the current of water in a stream, this variation could be measured experimentally by placing a paddle wheel in the stream as shown in the figure. Then the rotation of the paddle wheel would be a measure of the variation of the vector field. This can be done without getting wet by calculating a line integral around a typical closed curve C , as shown on the figure.

This integral can be used to define an average value of the variation (called **rot** or, more commonly, **curl**) over a surface S bounded by the curve C . The average **curl** is defined by

$$\langle \hat{\mathbf{n}} \cdot \mathbf{curl} \mathbf{E} \rangle_S = \frac{1}{S} \oint_C \mathbf{dr} \cdot \mathbf{E}, \quad (1.92)$$

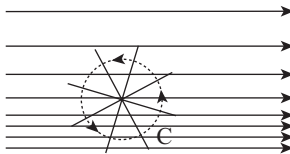


Figure 1.12: Velocity field with curl. The current increases going down on the figure causing the paddle wheel to rotate counterclockwise.

where $\hat{\mathbf{n}}$ is the unit vector normal to the surface S at any point. Note that, by this definition, the combination $S\langle\hat{\mathbf{n}}\cdot\mathbf{curl}\mathbf{E}\rangle_S$ does not depend on the shape of the surface S , but only on the bounding path C . Since the variation will be different in different directions, it is the average value of the normal component of **curl** that is defined by Eq. (1.92).

The positive sign for the direction of $\hat{\mathbf{n}}$ is taken by convention to be the **boreal** direction. That is, if integral around the contour C is taken in the direction of the rotation of the Earth, then the north pole is in the positive direction as shown on Fig. 1.13a. This is also stated as the **right-hand rule**: If the integral around the contour C is taken in the direction that the four fingers of the right-hand curl as they tend to close, then the right thumb points in the positive direction for $\hat{\mathbf{n}}$, as shown in Fig. 1.13b. This will be our general sign convention relating the direction of integration around a closed curve and the positive direction of the normal vector to any surface bounded by the curve.

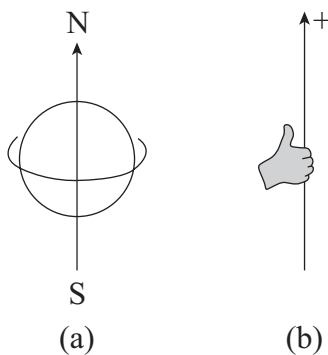


Figure 1.13: (a) Boreal direction on the globe. (b) Right-hand rule for positive direction.

The value of $\mathbf{curl}\mathbf{E}$ at a point can be defined by starting with a smooth surface through the point and taking the limit as the curve bounding the surface shrinks about the point, and the enclosed surface shrinks to zero area. This gives

the definition of **curl** at a point:

$$(\mathbf{curl} \mathbf{E})_n = \lim_{S \rightarrow 0} \frac{1}{S} \oint_C \mathbf{dr} \cdot \mathbf{E}. \quad (1.93)$$

As the curve C shrinks to a point, the smooth surface approaches its tangent plane at the point, and $(\mathbf{curl} \mathbf{E})_n$ in Eq. (1.92) represents the component of curl in the direction of the normal vector $\hat{\mathbf{n}}$ to the tangent plane.

With **curl** \mathbf{E} defined by Eq. (1.93), it is possible to find its specific form in Cartesian coordinates. To find **curl** \mathbf{E} at the point x, y, z , we consider an infinitesimal rectangle of dimension Δx by Δy parallel to the $x - y$ plane, as shown in Fig. 1.14. The line integral around the rectangle consists of four parts,

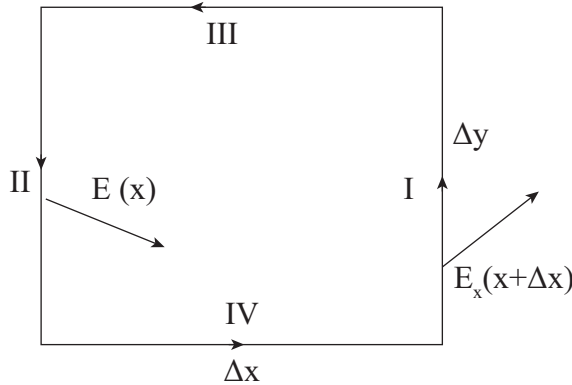


Figure 1.14: Differential surface for curl in Cartesian coordinates.

so

$$\lim_{S \rightarrow 0} \frac{1}{S} \oint_C \mathbf{dr} \cdot \mathbf{E} = I + II + III + IV. \quad (1.94)$$

As Δy approaches zero, we can make the replacement

$$\int_y^{y+\Delta y} f(y') dy' \rightarrow f(y) \Delta y, \quad (1.95)$$

provided that $f(y)$ is continuous at y . Then the contribution to Eq. (1.93) from sides I and II is

$$\begin{aligned} I + II &= \lim_{\Delta x, \Delta y \rightarrow 0} \frac{[E_y(x + \Delta x, y, z) \Delta y - E_y(x, y, z) \Delta y]}{\Delta x \Delta y} \\ &= \lim_{\Delta x \rightarrow 0} \frac{[E_y(x + \Delta x, y, z) - E_y(x, y, z)]}{\Delta x} \\ &= \partial_x E_y. \end{aligned} \quad (1.96)$$